Asymptotic behaviour of permutations avoiding generalized patterns

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Abstract

Visualizing permutations as labelled trees allows us to to specify restricted permutations, and to analyze their counting sequence. The asymptotic behaviour for permutations that avoid a given pattern is given by the Stanley-Wilf conjecture, which was proved by Marcus and Tardos in 2005. Another interesting question is the occurence of generalized patterns, i.e. patterns containing subwords. There are good asymptotic results for consecutive patterns and certain variations, but only specific results for patterns with subwords of length exactly 2. The goal of the project is to fully understand the analysis performed by Elizalde and Noy on such patterns, and to try to extend these results to other cases.

1 Introduction

Let $\sigma \in \mathcal{S}_k$ and $\pi \in \mathcal{S}_n$ be two permutations of length k and n, such that $k \leq n$. The permutation π is said to contain the permutation τ if and only if there exists a subsequence $1 \leq i_1 \leq i_2 \ldots \leq i_k \leq n$ such that the ordered sequence of elements $(\pi_{i_1}, \pi_{i_2}, \ldots, \pi_{i_k})$ is order-isomorphic to σ. Consequently, π avoids σ if no such subsequence exists.

Babson and Steingrimsson proposed a different definition of containment by introducing generalized patterns [1]. Their definition included the added restriction of having certain elements in the pattern appear consecutively in the permutation. The enumeration and asymptotic behaviour of permutations avoiding these generalized patterns was studied by Elizalde [2, 3], Noy [2] and Kitaev [4], and many of their results are included in this paper.

The paper is divided into seven sections, including the introduction. Section 2 introduces the concepts of generalized patterns and Wilf-equivalence, and also lays down the notation used in the paper. Section 3 works on the enumeration of certain classes of consecutive patterns, and puts forward the asymptotic results for the same. Section 4 focuses on the analysis of non-consecutive patterns, and includes some results that are used later in the paper. Section 5 studies permutations avoiding the pattern $12-34$, and Section 6 gives a similar analysis of permutations avoding $1-\sigma-k$. Section 7 considers the open questions in the field, and proposes a direction of further research.

2 Definitions and notations

2.1 Generalized patterns

For two positive integers $m \leq n$, let $\sigma = \sigma_1\sigma_2 \ldots \sigma_m \in \mathcal{S}_m$ and $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathcal{S}_n$ be two permutations, where $\mathcal{S}_k$ represents the symmetric group of order k. A generalized pattern is obtained from σ by denoting the permutation by $\sigma_1\epsilon_1\sigma_2\epsilon_2 \ldots \epsilon_{m-1}\sigma_m$, where each $\epsilon_j, j \in \{1, 2, \ldots, m-1\}$, is either empty or a dash$(-)$. Then, the permutation π is said to contain the *generalized pattern* [1] σ if there exist indices $i_1 < i_2 < \ldots < i_m$ marking positions of π such that:

(i) for each $j \in \{1, 2, \ldots, m-1\}$, if ϵ_j is empty, then $i_{j+1} = i_j + 1$.

(ii) the order-conserving reduction $\rho\left(\pi_{i_1}\pi_{i_2} \ldots \pi_{i_m}\right) = \sigma_1\sigma_2 \ldots \sigma_m$ holds true.

The subsequence $\pi_{i_1}\pi_{i_2} \ldots \pi_{i_m}$ is called an occurence of σ in π. The first condition imposes the restriction that the elements corresponding to adjacent numbers in the pattern must also be adjacent in the permutation π. The second condition is the classical pattern containment condition. For example, $\pi = 3542716$ contains the generalized pattern $\sigma = 12-4-3$ in the form $35-7-6$, but does not contain the pattern $\sigma = 12-43$.

If all the ϵ_j in a generalized pattern are dashes, then we get the definition of a classical pattern. The structure of permutations avoiding them is well studied, and the behaviour of the counting sequence of such permutations is given by the Stanley-Wilf conjecture. Consecutive patterns, in which there are no dashes, are of special interest because they provide a structure for defining classes of permutations that avoid other generalized patterns. The bivariate generating function for permutations that avoid a large class of consecutive patterns has been explored by Elizalde and Noy [2]. From these structures, it is easy to extend to permutations that avoid these patterns.

If σ is a generalized pattern, and $\mathcal{S}_n(\sigma)$ is the class of permutations of length n that avoid σ, then, fixing the notation $A_n(\sigma) = |\mathcal{S}_n(\sigma)|$, we define the exponential generating function [5] of patterns avoiding σ by:

$$A_\sigma(z) = \sum_{n \geq 0} A_n(\sigma)\frac{z^n}{n!}$$

2.2 Equivalence classes in patterns

Two classical patterns σ and τ are said to be equivalent if the bivariate generating function counting the number of occurences of σ and τ are the same [2]. Equivalence can also be established through these operations:

(i) Reversal: Given $\sigma = \sigma_1\sigma_2 \ldots \sigma_k$, the pattern $\bar{\sigma} = \sigma_k\sigma_{k-1} \ldots \sigma_2\sigma_1$ is equivalent to it.

(ii) Complementation: Given $\sigma = \sigma_1\sigma_2 \ldots \sigma_k$, the pattern $\tilde{\sigma} = (k+1-\sigma_1)(k+1-\sigma_2) \ldots \ldots (k+1-\sigma_k)$ is equivalent to it.

(iii) Inversion: Given σ and σ^{-1}, such that $\sigma^{-1}\sigma = \mathbf{I}$, where $\mathbf{I}$ is the identity map, both σ and σ^{-1} are equivalent.

The equivalence can be explained by the fact that if π contains σ, then $\bar{\pi}$ must contain $\bar{\sigma}$, $\tilde{\pi}$ must contain $\tilde{\sigma}$, and π^{-1} must contain σ^{-1}.

A special case of this is when two classes of permutations avoiding distinct patterns are counted by the coefficients of the same generating function. These classes are said to be *Wilf equivalent.*

2.3 Representations of permutations as increasing binary trees

A labelled binary tree in which the labels along any path from the root are increasing is called an increasing binary tree. Every permutation π can be split about its smallest element i into the set $\sigma i \tau$, where σ and τ are expanded permutations. Then, we can create a binay tree $T(\pi)$ with the root having label i, the left child labelled with $T(\sigma)$, and the right child with $T(\tau)$ [5]. The empty permutation yields the empty tree. This construction also establishes a bijection between the set of trees on n nodes and the number of permutations of size n.

2.4 Specification of increasing binary trees

Given the labelled class of increasing binary trees $\mathcal{T}$, we can recursively define its specification as follows:

$$\mathcal{T} = \mathcal{E} + \mathcal{Z}^{\Box} \star \mathcal{T} \star \mathcal{T}$$

Here, $\mathcal{E}$ is the empty element, and $\mathcal{Z}$ is the unit element. The boxed product restricts the root to have the smallest label. So, the exponential generating function [5] is:

$$\begin{aligned} T(z) &= 1 + \int_0^z T^2(t)\,dt \\ \Leftrightarrow T'(z) &= T^2(z) \qquad T(0) = 1 \\ \Leftrightarrow T(z) &= \frac{1}{1-z} \end{aligned}$$

The specification confirms our bijection with the permutations by giving us a counting sequence of n!. Such a specification allows the introduction of another variable that keeps track of the occurence of a certain pattern. In particular, this is a very useful construct for counting consecutive patterns.

3 Consecutive patterns

The distribution of consecutive patterns proves crucial to our understanding of the wider case of generalized patterns. Consecutive patterns are modelled by increasing binary trees, and this representation allows a direct application of the symbolic method to obtain the exponential generating function of the said class of permutations. Certain classes of consecutive patterns are easy to enumerate, and their asymptotic behaviour follows directly from their specification. Noy and Elizalde found the following results for two classes of patterns, whose proofs we have outlined.

Theorem 1. *[2] Given a consecutive increasing pattern $\sigma \in \mathcal{S}_{k+2}$, the bivariate generating function $P(z,u)$ where z counts the number of permutations containing the pattern, and u counts the number of occurences of σ, is given by $P(z,u) = 1/\omega(z,u)$, where:*

$$\omega^{(k+1)} + (1-u)\left(\omega^{(k)} + \omega^{(k-1)} + \ldots + \omega\right) = 0$$

with the initial conditions $\omega(0) = 1, \omega'(0) = -1, \omega^{(i)}(0) = 0 \,\forall\, i \in \{2,3,\ldots,k\}$

Proof. (Sketch) Let $\mathcal{P}$ be the class of all permutations, and let $\mathcal{K}_i$ be the class of permutations not beginning with $1,2,\ldots,k+2-i$. Then, we can specify the class of all permutations as follows:

$$\mathcal{P} = \mathcal{E} + \mathcal{Z}^{\Box} \star \mathcal{P} \star \left[\mathcal{K}_1 + \mu\left(\mathcal{P} - \mathcal{K}_1\right)\right]$$

i.e. the class of permutations consists of the empty permutation, or consists of elements that can be recursively broken up into an increasing binary tree. If so, the left child contains an element from the same class, and the right child contains either a class of permutations beginning with $1,2,\ldots k+1$, in which case we mark it by the element μ as an occurence of σ along with the label 1 on the root, or not, in which case we do not mark it.

Once we have this, we can define the following specification on each $\mathcal{K}_i$, where $i \in \{1,2,\ldots,k\}$, to get a set of $k+1$ specifications.

$$\mathcal{K}_i = \mathcal{E} + \mathcal{Z}^{\Box} \star \left(\mathcal{K}_i - \mathcal{E}\right) \star \left[\mathcal{K}_1 + \mu\left(\mathcal{P} - \mathcal{K}_1\right)\right] + \mathcal{Z}^{\Box} \star \mathcal{K}_{i+1}$$

Here, the first term is the empty permutation, the second counts every occurence of the permutation σ, and the last term exludes all the permutations that begin with the smallest label, followed by a labelling which is order isomorphic to $1,2,\ldots,k+1-i$.

Since each of these specifications translates to an integral equation, we can differentiate them with respect to z to get $k+1$ differential equations.

$$\begin{aligned}
P' &= P\left(K_1 + u\left(P - K_1\right)\right) \\
K_i' &= \left(K_i - 1\right)\left(K_1 + u\left(P - K_1\right)\right) + K_{i+1}
\end{aligned}$$

After making a suitable substitution, these can be solved with the initial conditions $P(0) = 1, K_i(0) = 1 \quad \forall\, i$ to get the specification mentioned. $\Box$

A very similar result exists for a consecutive pattern of the form $\sigma = 1,2,\ldots,a,\tau,a+1 \in \mathcal{S}_{k+2}$, where τ is an arbitrary permutation of $\{a+2, a+3, \ldots, k+2\}$. Since both a and τ can be varied, this covers a large class of permutations.

Theorem 2. *[2] For any consecutive pattern $\sigma = 1,2,\ldots,a,\tau,a+1 \in \mathcal{S}_{k+2}$, τ defined as above, the bivariate generating function $P(z,u)$ of permutations, where u marks the number of occurences of σ, is given by $P(z,u) = 1/\omega(z,u)$, where:*

$$\omega^{(a+1)} + (1-u)\frac{z^{k+1-a}}{(k+1-a)!}\omega' = 0$$

with the initial conditions $\omega(0) = 1, \omega'(0) = -1, \omega^{(i)}(0) = 0 \,\forall\, i \in \{2,3,\ldots,a\}$

3.1 Avoiding subwords of length three

3.1.1 Permutations avoiding 123

123 is an increasing subword. Using the equation stated in Theorem 1, we get the following specification for permutations without the pattern 123 by setting u to 0[2, 3].

$$P(z,0) = \frac{\sqrt{3}}{2} \frac{e^{z/2}}{\cos\left(\frac{\sqrt{3}z}{2} + \frac{\pi}{6}\right)}$$

As expected, the pattern 321 also gives us the same generating funtion, since it corresponds to a decreasing subword.

3.1.2 Permutations avoiding 132

The patterns $132, 231, 213, 312$ are all Wilf-equivalent. Using Theorem 2 and setting $u = 0$, we get the following solution[2, 3].

$$P(z,0) = \frac{1}{1 - \int_0^z e^{-t^2/2} dt} = \frac{1}{1 - \sqrt{\frac{\pi}{2}} erf\left(\frac{z}{\sqrt{2}}\right)}$$

3.2 Avoiding subwords of length 4

There are 7 Wilf classes among consecutive patterns of length 4. The classes for which we can get an explicit generating function for permutations avoiding the patterns are [2]:

$$\begin{aligned} A_{1234}(z) &= \frac{2}{e^{-z} + \cos z - \sin z} \\ A_{1432} &= \frac{1}{1 - \int_0^z e^{-z^3/6} dt} \\ A_{1243} &= \frac{1}{\omega} \end{aligned}$$

where ω is the solution to the differential equation $\omega''' + z\omega' = 0, \omega(0) = 1, \omega'(0) = -1, \omega''(0) = 0$.

3.3 Asymptotic enumeration for permutations avoiding consecutive patterns

The explicit generating functions obtained above allow us to use first degree asymptotics to estimate the growth of the coefficients. For patterns of length 3, the asymptotic behaviour of the coefficients is given as follows[2, 3]:

$$A_{123} \approx n! \gamma_1 \rho_1^{-n} \qquad\qquad A_{132} \approx n! \gamma_2 \rho_2^{-n}$$

where $\gamma_1 = \frac{3\sqrt{3}e^{\pi/3\sqrt{3}}}{2\pi}, \rho_1 = \frac{2\pi}{3\sqrt{3}}$, and $\gamma_2 = \frac{e^{\rho_2^2/2}}{\rho_2}, \rho_2 = \sqrt{2} erf^{-1}\left(\sqrt{\frac{2}{\pi}}\right)$.

For the known results of permutations avoiding patterns of length 4, the following relations hold [2].

$$\begin{array}{lll} A_{1234} \approx n!\gamma_1\rho_1^{-n} & \gamma_1 \approx 1.16063 & \rho_1 \approx 1.03842 \\ A_{1342} \approx n!\gamma_2\rho_2^{-n} & \gamma_2 \approx 1.83052 & \rho_2 \approx 1.04755 \\ A_{1243} \approx n!\gamma_3\rho_3^{-n} & \gamma_3 \approx 1.60433 & \rho_3 \approx 1.04944 \end{array}$$

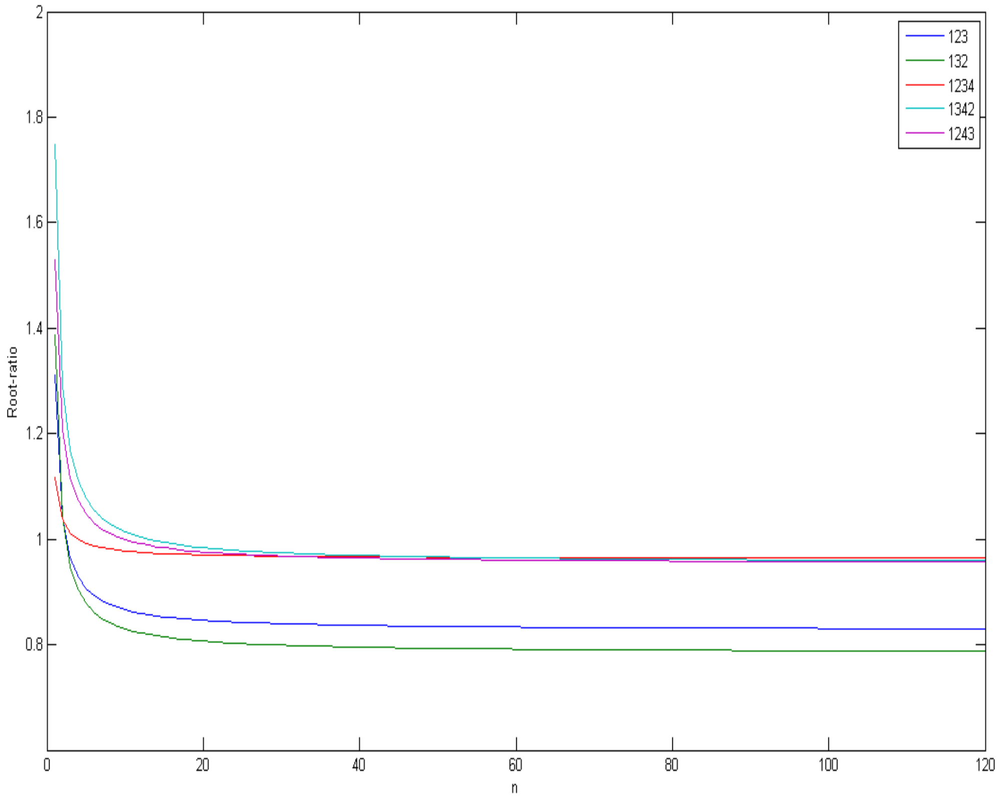

Figure 1: Plot of the value of $\sqrt[n]{A_n(\sigma)/n!}$, $n \leq 120$ for various consecutive patterns

The main observation here is that for each of these patterns, if we take $\lim_{n\to\infty} \sqrt[n]{\frac{A_n(\sigma)}{n!}}$, we get a finite, non-zero value. Elizalde proved this result in the following theorem.

Theorem 3. *[3] Let $k \geq 3$, and $\sigma \in \mathcal{S}_k$ be a consecutive pattern.*

(i) There exist constants $c, d \in (0,1)$ such that

$$c^n n! \; < \; A_n(\sigma) \; < \; d^n n! \qquad \forall\, n \geq k$$

(ii) There exists a constant $0 < w \leq 1$ *such that*

$$\lim_{n\to\infty} \sqrt[n]{\frac{A_n(\sigma)}{n!}} = w$$

Proof. Consider the number of permutations of length $m+n$ that avoid σ. We can choose any n consecutive elements of a permutation from this class, and those n elements, when placed in order, must avoid σ, as should the m elements that are left over. Since we can choose these elements in $\binom{m+n}{m,n}$ ways, we get an upper bound on the number of permutations of length $m+n$ that avoid σ.

$$A_{m+n}(\sigma) \leq A_m(\sigma) A_n(\sigma) \binom{m+n}{n}$$

Now, we can use induction on $n \geq k$, assuming that $A_n(\sigma) < d^n n!$ for some $0 < d < 1$. This gives us the following expression.

$$A_{m+n}(\sigma) < d^m m! d^n n! \binom{m+n}{n} = d^{m+n}(m+n)!$$

This proves the upper bound.

For the lower bound, we observe that any occurence of σ would automatically induce an occurence of a pattern of length 3 through reduction of the first three elements. Let us call this reduction τ. Thus, avoiding τ is a sufficient, but not necessary, condition for avoiding σ. So, $\mathcal{A}_n(\tau) \subseteq \mathcal{A}_n(\sigma)$, and $A_n(\tau) \leq A_n(\sigma)$. Since we know that $A_n(\tau)$ has first degree asymptotics of the form $n!\gamma\rho^{-n}$, we immediately have our lower bound.

$$c^n n! < A_n(132) \leq A_n(\sigma)$$

where $c = \frac{1}{\sqrt{2} erf^{-1}\left(\sqrt{\frac{2}{\pi}}\right)}$.

The second part of the proof involves the use of Fekete's lemma.

Lemma 1. *[6] Let* $f : \mathbb{N} \to \mathbb{N}$ *be a function for which* $f(m+n) \geq f(m) f(n)$ *for all* $m, n \in \mathbb{N}$. *Then,* $\lim_{n\to\infty} (f(n))^{1/n}$ *exists.*

Using the same upper bound on the number of permutations of length $m+n$ avoiding σ, we can rearrange the terms to get the following:

$$\frac{m!n!}{A_m(\sigma) A_n(\sigma)} \leq \frac{(m+n)!}{A_{m+n}(\sigma)}$$

By Fekete's lemma, then, $\lim_{n\to\infty} \sqrt[n]{\frac{A_n(\sigma)}{n!}}$ exists, and it is bounded by 1, according to the first part of the theorem. ☐

This is the major result for consecutive patterns, and forms a base for the study of simple non-consecutive generalized patterns.

4 Non-consecutive patterns

4.1 Classical patterns

Classical patterns are simply generalized patterns in which we have a dash between every two characters. The asymptotic behaviour of permutations avoiding classical patterns is described by the Stanley-Wilf conjecture, which was proved by Marcus and Tardos.

Theorem 4. *[7] For every classical pattern $\sigma = \sigma_1 - \sigma_2 - \ldots - \sigma_k$, there exists a constant λ which depends only on σ, such that:*

$$A_n(\sigma) < \lambda^n \qquad \forall\, n \geq 1$$

The theorem is equivalent to the statement that $\lim_{n\to\infty} \sqrt[n]{A_n(\sigma)}$ exists [8]. Determining this limit for various classes of patterns is a major thrust area, and this is known to be $(k-1)^2$ for the pattern $1 - 2 - \ldots - k$ [9], while it is 4 for patterns of length 3. The corresponding lower bound for classical patterns is given by the counting sequence of permutations avoiding the permutation $1 - 2 - 3$, i.e. which have at most one ascent. This is counted by the Catalan numbers [10].

4.2 Non-classical patterns

The class of permutations avoiding non-consecutive, non-classical patterns are, in general, hard to enumerate. These patterns can be divided into two categories according to the ease of analysis.

- Patterns having three consecutive elements, i.e. $\sigma = \ldots \sigma_{i-1}\sigma_i\sigma_{i+1}$.

  Theorem 5. *[3] Let $\sigma \in \mathcal{S}_k$ be a generalized pattern having three consecutive elements. Then, there exist constants $0 < c, d, < 1$ such that*

  $$c^n n! < A_n(\sigma) < d^n n! \qquad \forall\, n \geq k.$$

  Proof. Any occurence of the consecutive pattern obtained by removing all the dashes in σ must also indicate an occurence of σ. This gives us an upper bound on the number of permutations that avoid σ, and by Theorem 3, we have an upper bound on this. This proves the upper bound.

  For the lower bound, the three consecutive elements must correspong to some consecutive permutation on 3 elements, given by τ, and so $A_n(\sigma) \geq A_n(\tau)$. Since we have a bound for this in the form of $A_n(132) > c^n n!$, we can say that $A_n(\sigma) > c^n n!$. □

- Patterns having at least one occurence of two consecutive elements, but never having three consecutive elements. The asymptotic behaviour of permutations avoiding them is not known in general. There are a few cases that are well studied. The following result is by Claesson.

  Proposition 1. *[11] Let $\sigma \in \mathcal{S}_3$ be a generalized pattern with one dash.*

  (i) If $\sigma \in \{1-23, 3-21, 32-1, 12-3, 1-32, 23-1, 3-12, 21-3\}$, then $A_n(\sigma) = \mathbf{B}_n$.

  (ii) If $\sigma \in \{2-13, 2-32, 31-2, 13-2\}$, then $A_n(\sigma) = \mathbf{C}_n$.

  where $\mathbf{B}_n$ is the n^{th}-Bell number and $\mathbf{C}_n$ is the n^{th}-Catalan number.

4.2.1 Patterns of the form $1-\sigma$

The generating function of permutations avoiding patterns of the form $1-\sigma$, where σ is a consecutive pattern, can be described in terms of the generating function of permutations avoiding σ. The following results was given independently by Kitaev and Elizalde.

Proposition 2. ([3, 9]) *Let $\sigma = \sigma_1\sigma_2\ldots\sigma_{k-1} \in \mathcal{S}_{k-1}$ be a consecutive pattern, and let $1-\sigma$ be the generalized pattern $1-(\sigma_1+1)(\sigma_2+1)\ldots(\sigma_{k-1}+1)$. Then,*

$$A_{1-\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)dt\right)$$

Proof. Assume we have a permutation π. Let $m_1 > m_2 > \ldots > m_r$ be its left-to-right minima, where π_i is a left-to-right minima of π if, for every $j < i$, $\pi_i < \pi_j$. Thus, π can be written as $m_1w_1m_2w_2\ldots m_rw_r$, where each w_i consists of elements that are strictly greater than m_i. Also, by the definition of a left-to-right minimum, these elements must be greater than m_{i+1}. This decomposition is unique for every permutation, and imposes an order on the minima.

Now, if w_i contains σ, then, since m_i is strictly less than each element in w_i, this must correspond to an occurence of $1-\sigma$. Furthermore, σ must be strictly contained in a single block w_i, since it cannot contain the smallest label and all the elements that reduce to σ must be adjacent. Conversely, if each w_i avoids σ, there cannot be an occurence of $1-\sigma$. Thus, we get the following set construction.

$$\mathcal{A}_{1-\sigma} = SET\left(\mathcal{Z}^\Box \star \mathcal{A}_\sigma\right)$$

where the $\mathcal{Z}$ corresponds to the left-to-right minima m_i, and $\mathcal{A}_\sigma$ corresponds to an expanded permutation w_i avoiding σ. The set construction arises from the unique ordering of the minima, giving us a set of blocks m_iw_i. The box product confines the minimum label to the element corresponding to the minima. Thus, we get $A_{1-\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)dt\right)$. □

4.2.2 Patterns of the form $\sigma - k$

The generating function of patterns of the form $\sigma - k$ is the same as that of $1-\sigma$. The result was given by Kitaev. We shall derive the same result through a process inspired by Elizalde's proof for the pattern $1-\sigma$.

Proposition 3. *[4] Let $\sigma \in \mathcal{S}_{k-1}$ be a consecutive pattern. Then, the generating function for the generalized pattern $\sigma - k$ is given by:*

$$A_{\sigma-k}(z) = \exp\left(\int_0^z A_\sigma(t)dt\right)$$

Proof. Consider the permutation π. A right-to-left maximum of a permutation π is an element π_i such that $\pi_i > \pi_j$ for each $j > i$. Let us decompose π as $w_1M_1w_2M_2\ldots w_rM_r$, where $M_1 > M_2 > \ldots M_r$ are the right-to-left maxima of π, and each w_i consists of elements that are strictly less than M_{i-1}. Also, since $M_{i-1} > M_i$, by the definition of a right-to-left maximum, each element of w_i is less than M_i. This imposes an order on the occurence of the maxima, and gives us a unique decomposition for each permutation. An occurence of σ in w_i, along with M_i, would correspond to an occurence of $\sigma - k$ in π. Also, the elements corresponding to σ have to be adjacent, and must lie in the same block. Conversely, to avoid $1-\sigma$, each block w_i must avoid σ. This gives us the following construction.

$$\mathcal{A}_{1-\sigma} = SET\left(\mathcal{A}_\sigma \star \mathcal{Z}^{\blacksquare}\right)$$

where the $\mathcal{Z}$ corresponds to the right-to-left maxima M_i, and $\mathcal{A}_\sigma$ corresponds to an expanded permutation w_i avoiding σ. The set construction arises from the unique ordering of the maxima, giving us a set of blocks $w_i M_i$. The max-box product confines the maximum label to the element corresponding to the maxima. Thus, we get $A_{\sigma-k}(z) = \exp\left(\int_0^z A_\sigma(t)dt\right)$. □

4.2.3 Asymptotic behaviour of permutations avoiding $1-\sigma$ or $\sigma-k$

The case of permutations avoiding $1-\sigma$ or $\sigma-k$ is interesting because their asymptotic behaviour is equivalent to the behaviour of permutations avoiding σ.

Corollary 1. *[3] Let σ be a consecutive pattern. Then, defining the generalized patterns $1-\sigma$ and $\sigma-k$ as before,*

$$\lim_{n\to\infty}\left(\frac{A_n(1-\sigma)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{A_n(\sigma-k)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{A_n(\sigma)}{n!}\right)^{1/n} = w$$

where $0 < w \leq 1$, by Theorem 3.

Proof. This can be easily see from the fact that the exponential is an entire function, and the singularities of $A_\sigma(z)$ are preserved during the operation. Thus, $A_{1-\sigma}(z)$ and $A_{\sigma-k}(z)$ have the same radius of convergence as $A_\sigma(z)$, and thus, the coefficients grow in the same way. □

The symmetry of the patterns points towards a classification of generalized patterns of a certain length by the limiting constant. We shall expand on this in Section 6.

5 Permutations avoiding the pattern $12-34$

While we do not have an explicit expression for permutations avoiding the pattern $12-34$, Elizalde proved an upper and a lower bound for the counting sequence.

Proposition 4. *[3] For $k \geq 1$, define*

$$\begin{aligned}
b_k(z) &= \sum_{i=0}^{k}\binom{k}{i}^2\left[z + 2\left(\mathcal{H}_{k-i} - \mathcal{H}_i\right)\right]e^{iz} \\
c_k(z) &= \frac{e^{(k+1)z}}{k+1} - \sum_{i=0}^{k}\binom{k}{i}\binom{k+1}{i}\left[z + 2\left(\mathcal{H}_{k-i} - \mathcal{H}_i\right) + \frac{1}{k+1-i}\right]e^{iz} \\
S(z) &= \sum_{k\geq 1} b_k(z) + \sum_{k\geq 1} c_k(z)
\end{aligned}$$

where $\mathcal{H}_k$ is the harmonic sum up to k terms. Then,

$$[z^n]\, e^{S(z)} < [z^n]\, A_{12-34}(z) < [z^n]\, e^{S(z)+e^z+z-1}$$

Numerical results by Elizalde suggest that $\lim_{n\to\infty} \sqrt[n]{A_n(\sigma)/n!} = 0$, but this result has not been proved.

Proof. A permutation avoiding $12-34$ has no two ascents such that the the start of the second ascent is at a value higher than the end of the first. To construct this, we describe π as $B_0a_1B_2a_2\ldots$, where each a_i and the element preceding it form an ascent, B_0 is a non-empty decreasing word, and each B_i is composed of words $w_{i,0}w_{i,1}\ldots w_{i,r_i}$ that satisfy the following conditions.

(i) Each $w_{i,j}$ is a decreasing word.

(ii) If $r_i = 0$, then $w_{i,0}$ must be non-empty.

(iii) For each $j \geq 1$, $w_{i,j}$ is non-empty and its first element is larger than a_i.

(iv) The last element of each $w_{i,j}$ is less than a_i.

(v) The last element of w_{i,r_i} is less than a_{i+1}.

This ensures that the decomposition is unique.

To get an upper bound, we remove the restriction that the last element of w_{i,r_i} must be less than a_{i+1}. Thus, we also count some permutations that contain $12-34$, giving us an overestimate. Now, we get a set of blocks a_iB_i. In $B_i = w_{i,0}w_{i,1}\ldots w_{i,r_i}$, assume that $w_{i,0}$ is empty. If so, then consider the case when $r_i = 0$, i.e. the block only has a_i, and when $r_i = 1$, i.e. we get the block $a_iw_{i,1}$, and both the smallest and the largest labels must lie in $w_{i,1}$. Since $w_{i,1}$ is a decreasing word, it can be represented as a set, and the specification for the block becomes $\mathcal{Z} \star SET(\mathcal{Z})^{\boxtimes}$, where the double box product ensures that both the largest and the smallest label lie in $w_{i,1}$. The generating function for this block becomes:

$$\int_0^z \int_0^t u\frac{d^2e^u}{du^2}du\ dt = z(e^z-2)+z+2 = b_1(z)$$

For $r_i = 2$, we get the block $a_iw_{i,1}w_{1,2}$. The largest and the smallest labels may lie in either $w_{i,1}$ or $w_{i,2}$. We can split each $w_{i,j}$ into a part $w_{i,j}^+$, whose elements are all greater than a_i, and $w_{i,j}^-$, whose elements are less than a_i. Now, we separately consider the case in which the largest label lies in either $w_{i,1}^+$ or $w_{i,2}^+$, and in which the smallest label lies in either $w_{i,1}^-$ or $w_{i,1}^-$. This gives us 4 cases. If we had both labels in $w_{i,2}$, we would simply get the generating function $\int_0^z \int_0^t b_1(u)\frac{d^2e^u}{du^2}du\ dt$. But we can permute the different sections of both $w_{i,1}$ and $w_{i,2}$ without losing the structure specified. So, the generating function becomes:

$$4\int_0^z \int_0^t b_1(u)\frac{d^2e^u}{du^2}du\ dt = b_2(z)$$

Now, for any $b_{k-1}(z)$,

$$k^2\int_0^z \int_0^t b_{k-1}(u)e^u du\ dt\ = k^2\int_0^z \int_0^t \left(\sum_{i=0}^{k}\binom{k}{i}^2[u+2(\mathcal{H}_{k-i}-\mathcal{H}_i)]e^{iu}\right)e^u du\ dt = b_k(z)$$

Thus, $b_k(z)$ is the generating function of the block $a_iw_{i,1}w_{i,2}\ldots w_{i,k}$. In case $w_{i,0}$ is non-empty, the base case become the generating function of the block $a_iw_{i,0}$, which is $c_0(z) = e^z - 1 - z$, since we have a decreasing word of at least 2 elements. If $r_i = 1$, then we split $w_{i,1}$ into two parts as before. The smallest label can lie in either $w_{i,0}$ or $w_{i,1}^{-}$. We permute $w_{i,1}^{-}$ with $w_{i,0}$ to get another valid permutation. This gives us the following generating function.

$$2\int_0^z \int_0^t c_0(u) \frac{d^2 e^u}{du^2} du\, dt = c_1(z)$$

We can then use induction to get the following case for the generating function $c_k(z)$ of a block $a_i w_{i,0} w_{i,1} w_{i,2} \dots w_{i,k}$.

$$k(k+1)\int_0^z \int_0^t c_k(u)\, e^u du\, dt = c_k(z)$$

Our permutation π, with the restriction $w_{i,r_i} < a_{i+1}$ removed, and without counting the block B_0, is simply a set of blocks of arbitrary size in which $w_{i,0}$ may or may not be empty. Thus, we get the following generating function.

$$\exp\left(\sum_{k\geq 0} (b_k(z) + c_k(z))\right) = \exp\left(e^z - 1 + \sum_{k\geq 1} (b_k(z) + c_k(z))\right)$$

Now, the initial non-empty decreasing word B_0 will contribute e^z to this generating function to give us the desired upper bound.

For the lower bound, let us look at a block $a_i w_{i,0} w_{i,1} w_{i,2} \dots w_{i,r_i}$. If we split the last word into w_{i,r_i}^{+} and w_{i,r_i}^{-}, whose elements are greater and less than a_i respectively, then we can form a valid permutation $w_{i,r_i}^{-} a_i w_{i,0} w_{i,1} w_{i,2} \dots w_{i,r_i}^{+}$, since the last element of w_{i,r_i} is smaller than a_i. In effect, we are trying to impose the condition that the last element of $w_{i,r_{i-1}}$ must be less than a_i. Similarly, we can do the same if $w_{i,0}$ is empty. Clearly, this requires that $r_i \geq 1$. However, while this substitution gives us a unique permutation, not every permutation can be obtained through this process, since r_i may be 0 if $w_{i,0}$ is non-empty. Thus, we are undercounting, and will only get a lower bound. The n^{th} coefficient of the generating function for this class of permutations becomes:

$$[z^n] \exp\left(\sum_{k\geq 1} (b_k(z) + c_k(z))\right) < [z^n]\, A_{12-34}(z)$$

which completes our proof for the proposition. □

The decomposition given for $12 - 34$ also works for patterns of the form $12 - \sigma$, where σ is a consecutive pattern.

6 Permutations avoiding $1 - \sigma - k$

We saw that the class of permutations avoiding $1 - \sigma, \sigma - k$ and σ, where σ is a consecutive pattern of size $k - 1$, exhibit the same asymptotic behaviour. We now consider the class of permutations avoiding $1 - \sigma - k$, where σ is a consecutive pattern of size $k - 2$. The counting sequence of permutations avoiding the pattern $1 - \sigma - k$ is characterized by the following limits, again due to Elizalde.

Proposition 5. *[3] Let $\sigma \in \mathcal{S}_{k-2}$ be a consecutive pattern of length $k-2$, and let $1-\sigma-k$ be the pattern obtained by adding 1 to each element of σ. Let $C^{exp}(z)$ be the exponential generating function counting the Catalan numbers. Then*

$$[z^n] \int_0^z \int_0^u e^{y+2\int_0^y A_\sigma(t)dt} dydu < [z^n] A_{1-\sigma-k}(z) < [z^n] C^{exp}\left(\int_0^z A_\sigma(t)\,dt\right)$$

Proof. Consider the permutation π, and split it into blocks bookended by its left-to-right minima $\{a_1, a_2, \ldots, a_r\}$ and right-to-left-maxima $\{b_1, b_2, \ldots, b_s\}$. We can write π as $c_1 w_1 c_2 w_2 \ldots w_m c_m$, where each c_i is a left-to-right minimum or right-to-left maximum, and each w_i is a permutation of elements larger than the nearest left-to-right minimum to the left, and smaller than the nearest right-to-left maximum to the right. To avoid $1-\sigma-k$, each w_i must avoid σ.

To get the lower bound, consider the case in which all the minima occur before the maxima. So, $\pi = a_1 w_1 a_2 w_2 \ldots a_r w_r b_1 w_{r+1} b_2 \ldots w_{r+s-1} b_s$, where w_i is permutation avoiding σ, with all elements lying between a_i and b_1 when $1 \le i \le r$, and lying between a_r and b_i when $r+1 \le i \le r+s-1$, and w_r is a decreasing word. So, we have a set of blocks with the specification $\mathcal{Z}^{\square} \star \mathcal{A}_\sigma$, followed by a set of blocks of the type $\mathcal{A}_\sigma \star \mathcal{Z}^{\blacksquare}$, and an intermediate decreasing word specified by $SET(\mathcal{Z})$. Also, the smallest label lies in the first set, and the largest lies in the second set, giving us the following lower bound.

$$\int_0^z \int_0^u e^{\int_0^y A_\sigma(t)dt}\left(\frac{dy}{dy}\right) e^{\int_0^y A_\sigma(t)dt}\left(\frac{dy}{dy}\right) e^y dydu = \int_0^z \int_0^u e^{y+2\int_0^y A_\sigma(t)dt} dydu$$

which is the generating function of the lower bound.

To get the upper bound, consider the case when all the w_i's are empty. If every element is either a right-to-left maxima or a left-to-right minima, these permutations must avoid $1-2-3$, and are counted by the Catalan numbers [10]. If we replace each minima c_i by a block $c_i w_i$, the block w_i must avoid σ, and must be composed of elements greater than c_i. Alternately, if c_i is a maxima, we replace it by $w_i c_i$, where each element of w_i is less than c_i. Such a block corresponds to the generating function $\int_0^z A_\sigma(t)\,dt$, and we get an upper bound to the count by taking the composition, since we have eliminated the requirement of having no decreasing word after the last right-to-left maximum. Thus, the upper bound is given by the coefficients of $C^{exp}\left(\int_0^z A_\sigma(t)\,dt\right)$. ☐

Since we are only dealing in exponentials, we have the following result.

Corollary 2. *[3] For a consecutive pattern $\sigma \in \mathcal{S}_{k-2}$*

$$\lim_{n\to\infty}\left(\frac{A_n(1-\sigma-k)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{A_n(\sigma)}{n!}\right)^{1/n}$$

A second, direct result through Wilf equivalence is the following.

Corollary 3. *[3] For $\sigma, \tau \in \mathcal{S}_{k-2}$, if $A_\tau(z) = A_\sigma(z)$, then $A_{1-\tau-k}(z) = A_{1-\sigma-k}(z)$.*

In particular, we know the explicit results for the pattern $1-23-4$.

Proposition 6. *[3] For the generalized pattern $1-23-4$,*

$$[z^n]\,\frac{1}{2}\int_0^z e^{2e^t-2}dt - \frac{z}{2} < [z^n] A_{1-23-4}(z) < [z^n] C^{exp}(e^z - 1)$$

Also, the asymptotic result for the pattern is known.

Corollary 4. *[3] For the pattern* $1-23-4$

$$\lim_{n\to\infty}\left(\frac{A_n\left(1-23-4\right)}{n!}\right)^{1/n}=0$$

Proof. We can bound the coefficients of the Catalan generating function by the coefficients of e^{4z}. Since this converges for all z, $\lim_{n\to\infty}\left[z^n\right]e^{4z}=0$. This proves the result. □

7 Open problems

As seen in this paper, the asymptotic behaviour of generalized patterns has been hard to characterize, and we have mostly settled for bounds that we can work with. It might be useful to see if $\lim_{n\to\infty}\sqrt[n]{A_n\left(12-34\right)/n!}$ goes to 0 as conjectured [3]. The most obvious approach would be to loosen the upper bound. In general, the problem of finding $\lim_{n\to\infty}\sqrt[n]{A_n\left(\sigma\right)/n!}$ for generalized patterns, and $\lim_{n\to\infty}\sqrt[n]{A_n\left(\sigma\right)}$ for classical patterns is still under research.

One pattern that is currently being studied is $3-14-2$, since its subpatterns of length 3 correspond to patterns counted by the Catalan numbers, as given in Proposition 1. An analogous pattern is $3-21-4$, whose subpatterns of length 3 correspond to patterns counted by the Bell numbers. Also, neither of these patterns has subpatterns in the other class.

In [12], Elizalde proposed enumerative results for some generalized patterns by looking at rightward generating trees. The paper also considers permutations avoiding more than one pattern, as well as permutations avoiding barred patterns. Kitaev [4] studied a class of generalized patterns called *partially ordered generalized patterns*, and produced some general results that might be applied to the study of generalized patterns. More importantly, the definition of pattern containment offered in [4] can be used in formal languages.

A good research direction would be to study the property exhibited by the patterns $1-\sigma$,$\sigma-k$, $1-\sigma-k$ and σ, where σ is a consecutive pattern. Since these are counted by the same sequence, it might be worth asking the question if increasing patterns of the form $w_1-w_2-w_3$,w_1-w_2 and w_2-w_3 behave in a similar manner.

References

[1] E. Babson and E. Steingrımsson. Generalized permutation patterns and a classification of the mahonian statistics. *Sém. Lothar. Combin*, 44:B44b, 2000.

[2] S. Elizalde and M. Noy. Consecutive patterns in permutations. *Advances in Applied Mathematics*, 30(1-2):110–125, 2003.

[3] S. Elizalde. Asymptotic enumeration of permutations avoiding generalized patterns. *Advances in Applied Mathematics*, 36(2):138–155, 2006.

[4] S. Kitaev. Partially ordered generalized patterns. *Discrete Mathematics*, 298(1):212–229, 2005.

[5] P. Flajolet and R. Sedgewick. *Analytic combinatorics.* Cambridge Univ Pr, 2009.

[6] J.H. van Lint and R.M. Wilson. *A course in combinatorics.* Cambridge Univ Pr, 2001.

[7] A. Marcus and G. Tardos. Excluded permutation matrices and the stanley-wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004.

[8] R. Arratia. On the stanley-wilf conjecture for the number of permutations avoiding a given pattern. *Electron. J. Combin*, 6(1):1–4, 1999.

[9] A. Regev. Asymptotic values for degrees associated with strips of young diagrams. *Advances in Mathematics*, 41(2):115–136, 1981.

[10] D. Richards. Ballot sequences and restricted permutations. *Ars Combin*, 25(83):C86, 1988.

[11] A. Claesson. Generalized pattern avoidance. *European Journal of Combinatorics*, 22(7):961–971, 2001.

[12] S. Elizalde. Generating trees for permutations avoiding generalized patterns. *Annals of Combinatorics*, 11(3):435–458, 2007.

All symbolic calculations were done using Wolfram Alpha and MapleSoft. Graphs were plotted using MATLAB.